

# HIGHER ANALYTIC TORSION, POLYLOGARITHMS AND NORM COMPATIBLE ELEMENTS ON ABELIAN SCHEMES

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ABSTRACT. We give a simple axiomatic description of the degree 0 part of the polylogarithm on abelian schemes and show that its realisation in analytic Deligne cohomology can be described in terms of the Bismut-Köhler higher analytic torsion form of the Poincaré bundle.

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## INTRODUCTION

Norm compatible units play an important role in the arithmetic of cyclotomic fields and elliptic curves, especially in the context of Iwasawa theory and Euler systems. The norm compatible units in these cases are related to rich classical objects like polylogarithm functions and Eisenstein series.

The search for analogues in the case of abelian schemes was for a long time obstructed by the fact that the focus was narrowed on units rather than on classes in other  $K$ -groups. If one generalizes the question to classes in  $K_1$ , then two different constructions were given in [Kin99] and [MR]. The first construction in [Kin99] relies on the motivic polylogarithm and gives not only norm compatible classes but a whole series of classes satisfying a distribution relation. The second approach in [MR] depends on the arithmetic Riemann-Roch theorem applied to the Poincaré bundle and constructs a certain current, which turns out to be in the image of the regulator map from  $K_1$ .

It is a natural guess that these two approaches should be related or in fact more or less equivalent. In the following paper we show that this expectation is founded but in a rather ad hoc way. To explain our result, let us fix some notation. Let  $\pi : \mathcal{A} \rightarrow S$  be an abelian scheme of relative dimension  $g$ , let  $\epsilon : S \rightarrow \mathcal{A}$  be the zero section,  $N > 1$  an integer and let  $\mathcal{A}[N]$  be the finite group scheme of  $N$ -torsion points. Here  $S$  is smooth over a subfield of the complex numbers. Then the (zero step of the) motivic polylogarithm is a class in motivic cohomology

$$\text{pol}^0 \in H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g).$$

To describe it more precisely, consider the residue map along  $\mathcal{A}[N]$

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon(S), 0).$$

If we denote by  $(\cdot)^{(0)}$  the generalized eigenspace of  $\text{tr}_{[a]}$  with eigenvalue  $a^0 = 1$ , then the residue map becomes an isomorphism

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon(S), 0)^{(0)}$$

and  $\text{pol}^0$  is the unique element mapping to the fundamental class of  $\mathcal{A}[N] \setminus \epsilon(S)$ .

Similarly, the current  $\mathfrak{g}_{\mathcal{A}^\vee}$  on  $\mathcal{A}$  constructed in [MR] gives rise to a class in analytic Deligne cohomology

$$([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]} \in H_{D, \text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)),$$

which lies in the image of the regulator map

$$\text{cyc}_{\text{an}} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_{D, \text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)).$$

We prove:

**Theorem 1.** *We have*

$$-2 \cdot \text{cyc}_{\text{an}}(\text{pol}^0) = ([N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

Furthermore, the map

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \rightarrow H_{D, \text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

induced by  $\text{cyc}_{\text{an}}$  is injective.

(Theorem 1 is Theorem 4.1.2 below)

Unfortunately, the obvious strategy to prove this theorem (ie to show that the class in analytic Deligne cohomology described above satisfies the same axiomatic properties as the zero step of the polylogarithm) fails because analytic Deligne cohomology does not come with residue maps. Instead we proceed as follows. We first show that the theorem is true for an abelian scheme if it is true for one of its closed fibres. After that, using the compatibility of both the polylogarithm and the current under base change we show that it suffices to consider the universal abelian scheme over a suitable moduli space of abelian varieties. There we have a special fibre, which is a product of elliptic curves, where the theorem can be checked by direct computations.

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## 1. NOTATIONS

We fix a base scheme  $S$ , which is irreducible, smooth and quasi-projective over a field. This condition is necessary to apply the results of Deninger-Murre on the decomposition of the Chow-motive of an abelian scheme. We will work with an abelian scheme  $\pi : \mathcal{A} \rightarrow S$  of fixed relative dimension  $g$  with unit section  $\epsilon : S \rightarrow \mathcal{A}$ . For any variety over a field Soulé [Sou85] has defined motivic cohomology and homology

$$\begin{aligned} H_{\mathcal{M}}^i(X, j) &:= \text{Gr}_{\gamma}^j K_{2j-i}(X) \otimes \mathbb{Q} \\ H_i^{\mathcal{M}}(X, j) &:= \text{Gr}_j^{\gamma} K'_{i-2j}(X) \otimes \mathbb{Q}, \end{aligned}$$

which form a twisted Poincaré duality theory in the sense of Bloch-Ogus.

## 2. NORM COMPATIBLE ELEMENTS ON ABELIAN SCHEMES

**2.1. The trace operator.** Fix an integer  $a > 1$  prime to the characteristic of the ground field and let  $\mathcal{B} \rightarrow S$  be an abelian scheme. In the applications  $\mathcal{B}$  will be  $\mathcal{A}$  or  $\mathcal{A} \times_S \cdots \times_S \mathcal{A}$ . We consider open sub-schemes  $W \subset \mathcal{B}$  with the property that

$$j : [a]^{-1}(W) \subset W$$

is an open immersion, where  $[a] : \mathcal{B} \rightarrow \mathcal{B}$  is the  $a$ -multiplication on  $\mathcal{B}$ . As  $[a]$  is finite étale and  $j$  an open immersion, pull-back  $j^*$  and push-forward  $[a]_*$  are defined on motivic homology and cohomology.

**Definition 2.1.1.** For open sub-schemes  $W \subset \mathcal{B}$  as above, we define the *trace map* with respect to  $a$  by

$$\text{tr}_{[a]} : H_{\mathcal{M}}(W, *) \xrightarrow{j^*} H_{\mathcal{M}}([a]^{-1}(W), *) \xrightarrow{[a]_*} H_{\mathcal{M}}(W, *)$$

and

$$\mathrm{tr}_{[a]} : H_{\mathcal{M}}^{\cdot}(W, *) \xrightarrow{j^*} H_{\mathcal{M}}^{\cdot}([a]^{-1}(W), *) \xrightarrow{[a]_*} H_{\mathcal{M}}^{\cdot}(W, *).$$

For any integer  $r$ , we let

$$H_{\mathcal{M}}^{\cdot}(W, *)^{(r)} := \{\xi \in H_{\mathcal{M}}^{\cdot}(W, *) \mid (\mathrm{tr}_{[a]} - a^r \mathrm{id})^k \xi = 0 \text{ for some } k \geq 1\}$$

be the generalized eigenspace of  $\mathrm{tr}_{[a]}$  of *weight*  $r$ .

**Remark 2.1.2.** Of course, one can use any twisted Poincaré theory in the sense of Bloch-Ogus instead of motivic cohomology for the definition of the trace operator and its properties established below.

**Lemma 2.1.3.** *Suppose that  $\mathrm{tr}_{[a]}$  and  $\mathrm{tr}_{[b]}$  are defined on  $H_{\mathcal{M}}^{\cdot}(W, *)$  and  $H_{\mathcal{M}}^{\cdot}(W, *)$ , then the actions of  $\mathrm{tr}_{[a]}$  and  $\mathrm{tr}_{[b]}$  commute.*

*Proof.* This follows immediately from the diagram

$$\begin{array}{ccc} H_{\mathcal{M}}^{\cdot}([a]^{-1}W, *) & \xrightarrow{[a]_*} & H_{\mathcal{M}}^{\cdot}(W, *) \\ \downarrow & & \downarrow [a]_* \\ H_{\mathcal{M}}^{\cdot}([ab]^{-1}W, *) & \xrightarrow{[ab]_*} & H_{\mathcal{M}}^{\cdot}(W, *) \\ \uparrow & & \uparrow [b]_* \\ H_{\mathcal{M}}^{\cdot}([b]^{-1}W, *) & \xrightarrow{[b]_*} & H_{\mathcal{M}}^{\cdot}(W, *) \end{array}$$

and in exactly the same way for homology. □

The next lemma shows that the localization sequences are equivariant for the  $\mathrm{tr}_{[a]}$ -action in many situations.

**Lemma 2.1.4.** *Let  $W \subset \mathcal{B}$  be an open sub-scheme with  $[a]^{-1}W \subset W$  and  $Z \subset W$  be a closed sub-scheme. Define  $\tilde{Z}$  by the cartesian diagram*

$$\begin{array}{ccc} Z & \hookrightarrow & W \\ \downarrow & & \downarrow \\ \tilde{Z} & \hookrightarrow & [a]^{-1}W \end{array}$$

*and assume that  $[a](\tilde{Z}) = Z$ . Then the localization sequence*

$$\rightarrow H_{\mathcal{M}}^{\cdot}(Z, *) \rightarrow H_{\mathcal{M}}^{\cdot}(W, *) \rightarrow H_{\mathcal{M}}^{\cdot}(W \setminus Z, *) \rightarrow$$

*is equivariant for the  $\mathrm{tr}_{[a]}$ -action.*

*Proof.* As  $W$  is smooth and irreducible, the localization sequence in motivic homology is isomorphic to the localization sequence in cohomology with supports. This localization sequence is functorial

with respect to Cartesian diagrams of closed immersion by the usual Bloch-Ogus axioms, which leads to a commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow & H_{Z, \mathcal{M}}^\bullet(W, *) & \longrightarrow & H_{\mathcal{M}}^\bullet(W, *) & \longrightarrow & H_{\mathcal{M}}^\bullet(W \setminus Z, *) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_{\tilde{Z}, \mathcal{M}}^\bullet([a]^{-1}W, *) & \longrightarrow & H_{\mathcal{M}}^\bullet([a]^{-1}W, *) & \longrightarrow & H_{\mathcal{M}}^\bullet([a]^{-1}W \setminus \tilde{Z}, *) & \longrightarrow \\
& \cong \downarrow & & \cong \downarrow & & \cong \downarrow & \\
\longrightarrow & H_{2g-., \mathcal{M}}^\bullet(\tilde{Z}, *) & \longrightarrow & H_{2g-., \mathcal{M}}^\bullet([a]^{-1}W, *) & \longrightarrow & H_{2g-., \mathcal{M}}^\bullet([a]^{-1}W \setminus \tilde{Z}, *) & \longrightarrow
\end{array}$$

If we combine this with the functoriality of the homology localization sequence for the proper morphism  $[a]$ , we get the desired result.  $\square$

**2.2. Consequences of the motivic decomposition of the diagonal.** Let  $\Delta \subset \mathcal{A} \times_S \mathcal{A}$  be the relative diagonal and

$$\text{cl}(\Delta) \in H_{\mathcal{M}}^{2g}(\mathcal{A} \times_S \mathcal{A}, g)$$

its class in motivic cohomology. The main result by Deninger and Murre in [DM91, Theorem 3.1] implies that there is a decomposition

$$(2.2.1) \quad \text{cl}(\Delta) = \sum_{i=0}^{2g} \pi_i \in H_{\mathcal{M}}^{2g}(\mathcal{A} \times_S \mathcal{A}, g),$$

where the  $\pi_i$  are idempotents for the composition of correspondences and mutually commute. The  $\pi_i$  have the important property that for all integers  $a$

$$[a]_* \pi_i = a^{2g-i} \pi_i.$$

**Proposition 2.2.1.** *There is a decomposition into  $\text{tr}_{[a]}$ -eigenspaces*

$$H_{\mathcal{M}}^\bullet(\mathcal{A}, *) \cong \bigoplus_{r=0}^{2g} H_{\mathcal{M}}^\bullet(\mathcal{A}, *)^{(r)},$$

which is independent of  $a$ . Moreover,

$$H_{\mathcal{M}}^\bullet(\mathcal{A} \setminus \epsilon(S), *)^{(0)} = 0.$$

*Proof.* The first statement follows from the decomposition

$$H_{\mathcal{M}}^\bullet(\mathcal{A}, *) \cong \bigoplus_{i=0}^{2g} \pi_i H_{\mathcal{M}}^\bullet(\mathcal{A}, *),$$

which is independent of  $a$ . The second statement follows from the fact that  $\pi_{2g} H_{\mathcal{M}}^\bullet(\mathcal{A}, *) \cong \epsilon_* H_{\mathcal{M}}^{-2g}(S, * - g)$  and the equivariance of the localization sequence

$$\cdots \rightarrow H_{\mathcal{M}}^{-2g}(S, * - g) \xrightarrow{\epsilon_*} H_{\mathcal{M}}^\bullet(\mathcal{A}, *) \rightarrow H_{\mathcal{M}}^\bullet(\mathcal{A} \setminus \epsilon(S), *) \rightarrow \cdots$$

$\square$

The following simple corollary is basic for everything that follows:

**Corollary 2.2.2.** *Let  $N \geq 2$  and  $\mathcal{A}[N] \subset \mathcal{A}$  be the sub-scheme of  $N$ -torsion points, then*

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon(S), 0)^{(0)}.$$

*Proof.* This is a direct consequence of Proposition 2.2.1, the localization sequence

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \epsilon(S), g) \rightarrow H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \rightarrow H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon(S), 0) \rightarrow H_{\mathcal{M}}^{2g}(\mathcal{A} \setminus \epsilon(S), g)$$

and Lemma 2.1.4. □

Note that  $H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon(S), 0)^{(0)}$  is not zero. It contains the fundamental class of  $\mathcal{A}[N] \setminus \epsilon(S)$ .

### 3. THE POLYLOGARITHM ON ABELIAN SCHEMES

**3.1. A motivation from topology.** In this section we explain the topological polylogarithm. For more details and applications consult [BKL14].

Let  $X := \mathbb{C}^g / \Lambda$  a complex torus and consider the group ring  $\mathbb{Z}[\Gamma]$  with its standard action of  $\Gamma$  by  $\gamma(\gamma') := (\gamma + \gamma')$ . Let  $I$  be the augmentation ideal for  $\mathbb{Z}[\Gamma] \rightarrow \mathbb{Z}$ ,  $(\gamma) \mapsto 1$ . Define

$$\mathbb{Z}[[\Gamma]] := \varprojlim_n \mathbb{Z}[\Gamma] / I^{n+1}$$

and observe that  $I^n / I^{n+1} \cong \text{Sym}^n \Gamma$ . In fact, choosing a basis for  $\Gamma$ , one has  $\mathbb{Z}[\Gamma] \cong \mathbb{Z}[t_1, t_1^{-1}, \dots, t_{2g}, t_{2g}^{-1}]$  and  $I$  is the ideal  $(t_1 - 1, \dots, t_{2g} - 1)$ . The action of  $\Gamma$  on  $\mathbb{Z}[\Gamma]$  extends to  $\mathbb{Z}[[\Gamma]]$  and we denote by

$$\mathcal{L}og_{\mathbb{Z}}$$

the sheaf on  $X$  associated with the  $\Gamma$ -module  $\mathbb{Z}[[\Gamma]]$ . We also write

$$\mathcal{L}og_{\mathbb{Z}}^{(n)}$$

for the sheaf associated with the  $\Gamma$ -module  $\mathbb{Z}[[\Gamma]] / \widehat{I}^{n+1}$ , where  $\widehat{I}$  is the augmentation ideal of  $\mathbb{Z}[[\Gamma]]$ . One gets that  $\mathcal{L}og_{\mathbb{Z}} \cong \varprojlim_n \mathcal{L}og_{\mathbb{Z}}^{(n)}$ . Another description of  $\mathcal{L}og_{\mathbb{Z}}$  is as follows: Let  $p : \mathbb{C}^g \rightarrow X$  be the universal covering and consider  $p_! \mathbb{Z}$  on  $X$ . This sheaf is a  $\pi^* \mathbb{Z}[\Gamma]$ -module, where  $\pi : X \rightarrow \text{pt}$  is the structure map of  $X$ . Then

$$(3.1.1) \quad \mathcal{L}og_{\mathbb{Z}} \cong p_! \mathbb{Z} \otimes_{\pi^* \mathbb{Z}[\Gamma]} \pi^* \mathbb{Z}[[\Gamma]].$$

From this description, one sees without any effort that

$$(3.1.2) \quad R^i \pi_* \mathcal{L}og_{\mathbb{Z}} \cong \begin{cases} 0 & \text{if } i \neq 2g \\ \mathbb{Z} & \text{if } i = 2g, \end{cases}$$

because

$$R^i \pi_* \mathcal{L}og_{\mathbb{Z}} \cong H_c^i(X, p_! \mathbb{Z}) \otimes_{\pi^* \mathbb{Z}[\Gamma]} \pi^* \mathbb{Z}[[\Gamma]] \cong H_c^i(\mathbb{C}^g, \mathbb{Z}) \otimes_{\pi^* \mathbb{Z}[\Gamma]} \pi^* \mathbb{Z}[[\Gamma]]$$

as  $\mathbb{Z}[[\Gamma]]$  is a flat  $\mathbb{Z}[\Gamma]$ -module. Let  $\epsilon : \text{pt} \rightarrow X$  be the zero section, then the exact triangle

$$\epsilon_* \epsilon^! \mathcal{L}og_{\mathbb{Z}} \rightarrow \mathcal{L}og_{\mathbb{Z}} \rightarrow Rj_* j^* \mathcal{L}og_{\mathbb{Z}}$$

where  $j : X \setminus 0 \rightarrow X$  induces

$$\text{Ext}_X^{2g-1}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}}) \rightarrow \text{Ext}_{X \setminus 0}^{2g-1}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}}) \rightarrow \text{Hom}(\Gamma, I) \rightarrow \text{Ext}_X^{2g}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}})$$

where the Ext-groups are extension classes of local systems and  $\pi^* \Gamma$  is considered as trivial local system. From the above cohomology computation it follows that  $\text{Ext}_X^{2g-1}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}}) = 0$  and that  $\text{Ext}_X^{2g}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}}) \cong \text{Hom}(\Gamma, \mathbb{Z})$ . We get

$$0 \rightarrow \text{Ext}_{X \setminus 0}^{2g-1}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}}) \rightarrow \text{Hom}(\Gamma, \mathbb{Z}[[\Gamma]]) \rightarrow \text{Hom}(\Gamma, \mathbb{Z})$$

and the last map is easily seen to be induced by the augmentation of  $\mathbb{Z}[[\Gamma]]$ . Thus, we have an isomorphism

$$(3.1.3) \quad \text{Ext}_{X \setminus 0}^{2g-1}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}}) \cong \text{Hom}(\Gamma, \widehat{I}).$$

**Definition 3.1.1.** The *(topological) polylogarithm* is the class

$$\text{pol} \in \text{Ext}_{X \setminus 0}^{2g-1}(\pi^* \Gamma, \mathcal{L}og_{\mathbb{Z}})$$

that maps to the canonical inclusion  $\Gamma \subset \widehat{I}$  under the above isomorphism.

**3.2. Review of the logarithm sheaf.** Inspired by the above topological construction, one can define a logarithm sheaf in any reasonable sheaf theory, most notably for étale sheaves or Hodge-modules. This construction was first carried out in [Wil97] and generalizes the case of elliptic curves from [BL94]. The construction is very formal and does not need specific properties of the respective sheaf theory.

As in our main reference [Kin99], we want to define the logarithm sheaf at the same time for  $\ell$ -adic sheaves and for Hodge-modules over  $\mathbb{R}$ . In the Hodge-module setting we let  $F = \mathbb{R}$  and all varieties are considered over  $\mathbb{R}$ . Lisse sheaves are the ones where the underlying perverse sheaf is a local system placed in degree  $[-\text{dimension of the variety}]$ . We will consider these sheaves as sitting in degree 0 to have an easier comparison with the étale situation. In the  $\ell$ -adic setting we let  $F = \mathbb{Q}_{\ell}$  and a lisse sheaf is associated with a continuous representation of the étale fundamental group.

**Definition 3.2.1.** Let  $\mathcal{H} := (R^1 \pi_* F)^{\vee} = \text{Hom}_S(R^1 \pi_* F, F)$  be the dual of the first relative cohomology of  $\pi : \mathcal{A} \rightarrow S$ .

The Leray spectral sequence induces a short exact sequence

$$0 \rightarrow \text{Ext}_S^1(F, \mathcal{H}) \xrightarrow{\pi^*} \text{Ext}_{\mathcal{A}}^1(F, \pi^* \mathcal{H}) \rightarrow \text{Hom}_S(F, R^1 \pi_* F \otimes \mathcal{H}) \rightarrow 0,$$

which is exact and split because  $\pi$  has the section  $\epsilon : S \rightarrow \mathcal{A}$ . Note that

$$\text{Hom}_S(F, R^1 \pi_* F \otimes \mathcal{H}) \cong \text{Hom}_S(\mathcal{H}, \mathcal{H}).$$

**Definition 3.2.2.** Let  $\mathcal{L}og^{(1)} \in \text{Ext}_{\mathcal{A}}^1(F, \pi^* \mathcal{H})$  be the unique class, which maps to  $\text{id}_{\mathcal{H}} \in \text{Hom}_S(\mathcal{H}, \mathcal{H})$  and such that  $\epsilon^* \mathcal{L}og^{(1)}$  splits. We write also  $\mathcal{L}og^{(1)}$  for the lisse sheaf representing this class.

By definition  $\mathcal{L}og^{(1)}$  sits in an exact sequence

$$0 \rightarrow \pi^* \mathcal{H} \rightarrow \mathcal{L}og^{(1)} \rightarrow F \rightarrow 0.$$

We define

$$\mathcal{L}og^{(n)} := \text{Sym}^n \mathcal{L}og^{(1)}$$

so that we have morphisms  $\mathcal{L}og^{(n)} \rightarrow \mathcal{L}og^{(n-1)}$  induced by  $\mathcal{L}og^{(1)} \rightarrow F$ . We write  $\mathcal{L}og$  for the projective system  $(\mathcal{L}og^{(n)})_n$ . Note that  $\mathcal{L}og$  is a pro-unipotent sheaf, which is a successive extension of  $\text{Sym}^k \mathcal{H}$ .

Let  $\psi : \mathcal{A} \rightarrow \mathcal{B}$  be an isogeny. Then  $\psi$  induces an isomorphism  $\mathcal{H}_{\mathcal{A}} \cong \psi^* \mathcal{H}_{\mathcal{B}}$  and hence an isomorphism  $\mathcal{L}og_{\mathcal{A}}^{(1)} \cong \psi^* \mathcal{L}og_{\mathcal{B}}^{(1)}$  or more generally

$$(3.2.1) \quad \mathcal{L}og_{\mathcal{A}} \cong \psi^* \mathcal{L}og_{\mathcal{B}}.$$

For  $t \in \ker \psi(S)$ , we get  $t^* \mathcal{L}og_{\mathcal{A}} \cong \epsilon_{\mathcal{A}}^* \mathcal{L}og_{\mathcal{B}}$ , which induces an isomorphism

$$t^* \mathcal{L}og_{\mathcal{A}} \cong \epsilon_{\mathcal{A}}^* \mathcal{L}og_{\mathcal{A}} \cong \prod_{n \geq 0} \text{Sym}^n \mathcal{H}_{\mathcal{A}}.$$

As in the topological case one can compute the cohomology of  $\mathcal{L}og$ .

**Proposition 3.2.3** ([Kin99], Proposition 1.1.3). *For the higher direct images of  $\mathcal{L}og^{(n)}$  one has*

$$R^{2g} \pi_* \mathcal{L}og^{(n)} \cong R^{2g} \pi_* \mathcal{L}og^{(n-1)} \cong \dots \cong R^{2g} \pi_* F \cong F(-g)$$

and for  $i < 2g$  the maps  $\mathcal{L}og^{(n)} \rightarrow \mathcal{L}og^{(n-1)}$  induce the zero map

$$R^i \pi_* \mathcal{L}og^{(n)} \cong R^i \pi_* \mathcal{L}og^{(n-1)}$$

for all  $n$ .

Let us define

$$H^j(\mathcal{A}, \mathcal{L}og(g)) := \varprojlim_n H^j(\mathcal{A}, \mathcal{L}og^{(n)}(g)),$$

then the Proposition 3.2.3 and the Leray spectral sequence for  $R\pi_*$  imply that

$$(3.2.2) \quad H^j(\mathcal{A}, \mathcal{L}og(g)) \cong \begin{cases} 0 & \text{if } j < 2g \\ H^0(S, F) & \text{if } j = 2g. \end{cases}$$



**3.3. The polylogarithm.** Consider the exact triangle for the open immersion  $j : \mathcal{A} \setminus \mathcal{A}[N] \hookrightarrow \mathcal{A}$  and  $\iota : \mathcal{A}[N] \hookrightarrow \mathcal{A}$

$$\iota_* \iota^! \mathcal{L}og \rightarrow \mathcal{L}og \rightarrow Rj_* j^* \mathcal{L}og,$$

then we get a localization sequence

$$H^{2g-1}(\mathcal{A}, \mathcal{L}og(g)) \rightarrow H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}og(g)) \rightarrow H^0(\mathcal{A}[N], \iota^* \mathcal{L}og) \rightarrow H^{2g}(\mathcal{A}, \mathcal{L}og(g)).$$

**Corollary 3.3.1.** *Let  $\mathcal{L}og[\mathcal{A}[N]]^0 := \ker(\iota_* \iota^* \mathcal{L}og \rightarrow F)$  be the kernel of the map induced by the augmentation  $\mathcal{L}og \rightarrow F$ . Then the localization sequence induces an isomorphism*

$$H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}og(g)) \cong H^0(S, \mathcal{L}og[\mathcal{A}[N]]^0).$$

*Proof.* From Equation (3.2.2) we get that the localization sequence gives

$$0 \rightarrow H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}og(g)) \rightarrow H^0(\mathcal{A}[N], \iota^* \mathcal{L}og) \rightarrow H^0(S, F)$$

and the last map is induced by the augmentation  $\mathcal{L}og \rightarrow F$ . □

Denote by  $H^0(\mathcal{A}[N], F)^0$  the kernel of the trace map

$$H^0(\mathcal{A}[N], F)^0 := \ker(H^0(\mathcal{A}[N], F) \rightarrow H^0(S, F)).$$

By Equation (3.2.1) we have an inclusion

$$H^0(\mathcal{A}[N], F)^0 \subset H^0(S, \mathcal{L}og[\mathcal{A}[N]]^0).$$

It is convenient to identify

$$(3.3.1) \quad H^0(\mathcal{A}[N], F)^0 \cong H^0(\mathcal{A}[N] \setminus \epsilon(S), F)$$

via the restriction map.

**Definition 3.3.2.** For each  $\varphi \in H^0(\mathcal{A}[N] \setminus \epsilon(S), F)$  we let

$$\text{pol}_\varphi \in H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}og(g))$$

be the class, which maps to  $\varphi$  under the isomorphism in Corollary 3.3.1. We let

$$\text{pol}_\varphi^n \in H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}og^{(n)}(g))$$

be the image under the canonical map  $\mathcal{L}og \rightarrow \mathcal{L}og^{(n)}$ .

If we want to specify the theory of sheaves we working with, we write

$$\text{pol}_{\mathcal{H}, \varphi} \in H_{\mathcal{H}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}og(g))$$

for the absolute Hodge and

$$\text{pol}_{\text{et}, \varphi} \in H_{\text{et}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}og(g))$$

for the  $\ell$ -adic polylogarithm.

**3.4. Norm compatibility of the polylogarithm.** Let  $a \geq 2$  be an integer and consider the  $[a]$ -multiplication  $[a] : \mathcal{A} \rightarrow \mathcal{A}$ . We define an endomorphism  $\mathrm{tr}_{[a]}$  of  $H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\mathrm{og}(g))$  as follows:

$$\begin{aligned} \mathrm{tr}_{[a]} : H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\mathrm{og}(g)) &\rightarrow H^{2g-1}([a]^{-1}(\mathcal{A} \setminus \mathcal{A}[N]), \mathcal{L}\mathrm{og}(g)) \rightarrow \\ &\rightarrow H^{2g-1}([a]^{-1}(\mathcal{A} \setminus \mathcal{A}[N]), [a]^* \mathcal{L}\mathrm{og}(g)) \xrightarrow{\mathrm{trace}_{[a]}} H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\mathrm{og}(g)), \end{aligned}$$

where  $\mathrm{trace}_{[a]}$  is the trace map relative to the finite morphism  $[a]$ .

**Proposition 3.4.1.** *Suppose that  $(a, N) = 1$  and let  $[a]_* \varphi$  be the image of  $\varphi \in H^0(\mathcal{A}[N] \setminus \epsilon(S), F)$  under the finite map  $[a]$ . Then one has*

$$\mathrm{tr}_{[a]} \mathrm{pol}_\varphi = \mathrm{pol}_{[a]_* \varphi}.$$

*In particular, for  $a \equiv 1 \pmod{N}$  one gets  $\mathrm{tr}_{[a]} \mathrm{pol}_\varphi = \mathrm{pol}_\varphi$  and  $\mathrm{pol}_\varphi$  is norm-compatible.*

*Proof.* As the trace map is compatible with residues, we have a commutative diagram

$$\begin{array}{ccc} H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\mathrm{og}(g)) & \xrightarrow{\mathrm{res}} & H^0(S, \mathcal{L}\mathrm{og}[\mathcal{A}[N]]^0) \\ \mathrm{tr}_{[a]} \downarrow & & \downarrow \mathrm{tr}_{[a]} \\ H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\mathrm{og}(g)) & \xrightarrow{\mathrm{res}} & H^0(S, \mathcal{L}\mathrm{og}[\mathcal{A}[N]]^0). \end{array}$$

The map  $\mathrm{tr}_{[a]}$  induces on  $H^0(\mathcal{A}[N], F)^0 \subset H^0(S, \mathcal{L}\mathrm{og}[\mathcal{A}[N]]^0)$  the map  $[a]_* : H^0(\mathcal{A}[N], F)^0 \rightarrow H^0(\mathcal{A}[N], F)^0$ . The result follows from the definition of the polylogarithm.  $\square$

**Corollary 3.4.2.** *Let  $\mathrm{pol}_\varphi^0 \in H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], F(g))$  be the image of  $\mathrm{pol}_\varphi$  under the morphism*

$$H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\mathrm{og}(g)) \rightarrow H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], F(g))$$

*induced by the augmentation  $\mathcal{L}\mathrm{og} \rightarrow F$ . Then*

$$\mathrm{pol}_\varphi^0 \in H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], F(g))^{(0)}$$

*is in the generalized 0 eigenspace of  $\mathrm{tr}_{[a]}$ .*

*Proof.* This is clear from Proposition 3.4.1.  $\square$

**3.5. A motivic construction of the polylogarithm.** The motivic construction presented here is similar to the one in [Kin99], except that we consider here the variant of the polylogarithm explained in 3.3.2. We will focus on the differences and refer to [Kin99] whenever possible.

**Remark 3.5.1.** All constructions in this section work without any changes in any twisted Bloch-Ogus cohomology theory. To fix ideas, and because it is the "universal" case, we wrote everything for motivic cohomology.

Let us recall some notations from [Kin99]. Define

$$U := (\mathcal{A} \setminus \epsilon(S)) \times_S (\mathcal{A} \setminus \epsilon(S))$$

considered with the second projection  $p : U \rightarrow \mathcal{A} \setminus \epsilon(S)$  as a scheme over  $\mathcal{A} \setminus \epsilon(S)$ . Let  $V := U \setminus \Delta$  be the complement of the diagonal and set

$$U^n := U \times_{\mathcal{A} \setminus \epsilon(S)} \cdots \times_{\mathcal{A} \setminus \epsilon(S)} U \text{ } n\text{-times}$$

$$V^n := V \times_{\mathcal{A} \setminus \epsilon(S)} \cdots \times_{\mathcal{A} \setminus \epsilon(S)} V \text{ } n\text{-times}.$$

More generally, we let for  $I \subset \{1, \dots, n\}$

$$V^I := \{(u_1, \dots, u_n) \in U^n \mid u_i \in V \text{ if } i \in I \text{ and } u_i \in \Delta \text{ if } i \notin I\}.$$

This gives a stratification  $U^n = \bigcup_I V^I$ . Denote by  $\Sigma_n$  the permutation group of  $\{1, \dots, n\}$  and let  $\text{sgn}_n$  denote the sign character. For any  $\mathbb{Q}$ -vector space  $H$  with  $\Sigma_n$  action, we denote by  $H_{\text{sgn}_n}$  the  $\text{sgn}_n$  eigenspace.

The fundamental result for the construction is:

**Proposition 3.5.2.** *[Kin99] Corollary 2.1.4] There is a long exact sequence*

$$\rightarrow H_{\mathcal{M}}(U^n, *)_{\text{sgn}_n} \rightarrow H_{\mathcal{M}}(V^n, *)_{\text{sgn}_n} \xrightarrow{\text{res}} H_{\mathcal{M}}^{-2g+1}(V^{n-1}, * - g)_{\text{sgn}_{n-1}} \rightarrow,$$

where the residue map is taken along the  $n$ -th variable and which is equivariant for the  $\text{tr}_{[a]}$  action for any integer  $a$ .

The schemes  $U^n$  and  $V^n$  are considered over  $\mathcal{A} \setminus \epsilon(S)$  and we consider the base change to  $\mathcal{A} \setminus \mathcal{A}[N] \subset \mathcal{A} \setminus \epsilon(S)$ :

$$U_{\mathcal{A} \setminus \mathcal{A}[N]}^n := U^n \times_{\mathcal{A} \setminus \epsilon(S)} \mathcal{A} \setminus \mathcal{A}[N]$$

$$V_{\mathcal{A} \setminus \mathcal{A}[N]}^n := V^n \times_{\mathcal{A} \setminus \epsilon(S)} \mathcal{A} \setminus \mathcal{A}[N].$$

Note that the base change is compatible with the  $\Sigma_n$  action, so that the same proof as for [Kin99, Corollary 2.1.4] shows that there is also a long exact sequence

$$\rightarrow H_{\mathcal{M}}(U_{\mathcal{A} \setminus \mathcal{A}[N]}^n, *)_{\text{sgn}_n} \rightarrow H_{\mathcal{M}}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, *)_{\text{sgn}_n} \xrightarrow{\text{res}} H_{\mathcal{M}}^{-2g+1}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^{n-1}, * - g)_{\text{sgn}_{n-1}} \rightarrow.$$

This sequence is still equivariant for the  $\text{tr}_{[a]}$ -action. Now we use the fact that

$$(3.5.1) \quad H_{\mathcal{M}}(U^n, *)^{(0)} = 0,$$

which is an easy consequence of Proposition 2.2.1 by induction (see also [Kin99, Theorem 2.2.3]). Note that the vanishing does not depend on the  $a$  chosen to define  $\text{tr}_{[a]}$ . If we combine this with the long exact sequence in the proposition we get:

**Corollary 3.5.3.** *The residue maps induce isomorphisms*

$$\begin{aligned} H_{\mathcal{M}}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, *)_{\text{sgn}_n}^{(0)} &\xrightarrow{\cong} H_{\mathcal{M}}^{-2g+1}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^{n-1}, * - g)_{\text{sgn}_{n-1}}^{(0)} \xrightarrow{\cong} \cdots \\ &\cdots \xrightarrow{\cong} H_{\mathcal{M}}^{-(n+1)(2g-1)}(\mathcal{A}[N] \setminus \epsilon(S), * - (n+1)g)^{(0)}, \end{aligned}$$

which do not depend on the integer  $a$  used to define the operator  $\text{tr}_{[a]}$ .

We can now define the motivic polylogarithm.

**Definition 3.5.4.** For any  $\varphi \in H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon(S), 0)^{(0)}$  define the *motivic polylogarithm*

$$\text{pol}_{\mathcal{M}, \varphi}^n \in H_{\mathcal{M}}^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g)_{\text{sgn}_n}^{(0)}$$

to be the class, which maps to  $\varphi$  under the isomorphisms in Corollary 3.5.3.

From this characterization we get immediately:

**Proposition 3.5.5.** *The polylogarithm is compatible with base change.*

*Proof.* By [Dég04] the Gysin sequence is compatible with base change.  $\square$

**3.6. Comparison with the realizations of the polylogarithm.** In this section we relate the motivic polylogarithm  $\text{pol}_{\mathcal{M}, \varphi}$  from Definition 3.5.4 via the regulator maps to the  $\text{pol}_{\varphi}$  as in Definition 3.3.2.

Consider the regulator maps into absolute Hodge

$$r_{\mathcal{H}} : H_{\mathcal{M}}^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g) \rightarrow H_{\mathcal{H}}^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g)$$

and to  $\ell$ -adic cohomology

$$r_{\ell} : H_{\mathcal{M}}^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g) \rightarrow H_{\text{et}}^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g).$$

As in Section 3.2 we will treat the absolute Hodge and the  $\ell$ -adic case simultaneously. We start by identifying  $H^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g)_{\text{sgn}_n}^{(0)}$ .

**Proposition 3.6.1.** *One has an isomorphism*

$$H^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g)_{\text{sgn}_n}^{(0)} \cong H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\text{og}^{(n)}(g))^{(0)}$$

and a commutative diagram

$$\begin{array}{ccc} H^{(2g-1)(n+1)}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^n, (n+1)g)_{\text{sgn}_n}^{(0)} & \xrightarrow{\text{res}} & H^{(2g-1)n}(V_{\mathcal{A} \setminus \mathcal{A}[N]}^{n-1}, ng)_{\text{sgn}_{n-1}}^{(0)} \\ \cong \downarrow & & \downarrow \cong \\ H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\text{og}^{(n)}(g))^{(0)} & \longrightarrow & H^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], \mathcal{L}\text{og}^{(n-1)}(g))^{(0)}, \end{array}$$

where the lower horizontal map is induced by  $\mathcal{L}\text{og}^{(n)} \rightarrow \mathcal{L}\text{og}^{(n-1)}$ .

*Proof.* This is essentially proven in [Kin99, Proposition 2.3.1] but for the weight 1 parts.

Let  $\overline{U} := (\mathcal{A} \setminus \epsilon(S)) \times \mathcal{A}$  and  $\overline{V} := \overline{U} \setminus \Delta$ . We denote by  $\overline{p}$  the second projection. We also let  $\tilde{V}^n := V_{\mathcal{A} \setminus \mathcal{A}[N]}^n$  to shorten notation. It is shown in the first part of the proof of [Kin99, Proposition 2.3.1] that

$$R^i \overline{p}_* F_{\overline{V}}(g) \cong \begin{cases} \pi^* R^i \pi_* F(g) & i < 2g-1 \\ \mathcal{L}\text{og}^{(1)} & i = 2g-1 \\ 0 & i = 2g. \end{cases}$$

It follows that  $R^{2g-1}p_*F_{\tilde{V}}$  is isomorphic to the restriction of  $\mathcal{L}\text{og}^{(1)}$  to  $\mathcal{A}\backslash\mathcal{A}[N]$ .

Let  $p_{V^n} : \tilde{V}^n \rightarrow \mathcal{A}\backslash\mathcal{A}[N]$  be the structure map. As the  $R^i p_{V^n,*}F_{\tilde{V}^n}$  are all lisse sheaves, we can take the dual of the Künneth formula for  $Rp_{V^n,!}$  to get

$$R^i p_{V^n,*}F_{\tilde{V}^n} \cong \bigoplus_{i_1+\dots+i_n=i} R^{i_1}p_*F_{\tilde{V}} \otimes \dots \otimes R^{i_n}p_*F_{\tilde{V}}.$$

In particular,  $R^{(2g-1)n}p_{V^n,*}F_{\tilde{V}^n} \cong (\mathcal{L}\text{og}^{(1)})^{\otimes n}$  on  $\mathcal{A}\backslash\mathcal{A}[N]$ . Let  $\tilde{\pi} : \mathcal{A}\backslash\mathcal{A}[N] \rightarrow S$  be the structure map. Note that the projection formula for  $R\tilde{\pi}_!$  gives for lisse sheaves a projection formula for  $R\tilde{\pi}_*$ . From the exact sequence

$$0 \rightarrow \pi^*\mathcal{H} \rightarrow \mathcal{L}\text{og}^{(1)} \rightarrow F \rightarrow 0$$

we see that  $\text{tr}_{[a]}$  acts with weights  $\geq 0$  on  $R^i\tilde{\pi}_*\mathcal{L}\text{og}^{(1)}$ . We claim that

$$(3.6.1) \quad H^*(\mathcal{A}\backslash\mathcal{A}[N], R^{i_1}p_*F_{\tilde{V}} \otimes \dots \otimes R^{i_n}p_*F_{\tilde{V}}(gn))^{(0)} = 0$$

whenever  $i_1 + \dots + i_n < (2g-1)n$ . If this is the case, at least one factor in the tensor product is of the form  $\pi^*R^{i_j}\pi_*F$  with  $i_j < 2g-1$ . Without loss of generality, we may assume  $j=1$ . We get

$$\begin{aligned} R^k\tilde{\pi}_*(R^{i_1}p_*F_{\tilde{V}} \otimes \dots \otimes R^{i_n}p_*F_{\tilde{V}}) &\cong \\ &\cong R^{i_1}\pi_*F \otimes R^k\tilde{\pi}_*(R^{i_2}p_*F_{\tilde{V}} \otimes \dots \otimes R^{i_n}p_*F_{\tilde{V}}). \end{aligned}$$

The trace operator  $\text{tr}_{[a]}$  acts on  $R^{i_1}\pi_*F$  with weight  $\geq 2$  on  $(R^{i_2}p_*F_{V_{\mathcal{A}\backslash\mathcal{A}[N]}} \otimes \dots \otimes R^{i_n}p_*F_{V_{\mathcal{A}\backslash\mathcal{A}[N]}})$  with some weight  $\geq 0$ . This gives the vanishing in Equation (3.6.1). We get

$$H^{(2g-1)(n+1)}(\tilde{V}^n, (n+1)g)_{\text{sgn}_n}^{(0)} \cong H^{2g-1}(\mathcal{A}\backslash\mathcal{A}[N], \text{Sym}^n \mathcal{L}\text{og}^{(1)}),$$

which gives the first claim of the proposition. The compatibility of the residue with  $\mathcal{L}\text{og}^{(n)} \rightarrow \mathcal{L}\text{og}^{(n-1)}$  follows from the commutative diagram

$$\begin{array}{ccccc} H^{(2g-1)(n+1)}(\tilde{V}^n, (n+1)g)_{\text{sgn}_n}^{(0)} & \longrightarrow & H^{(2g-1)(n+1)}(\tilde{V}^n, (n+1)g)_{\text{sgn}_n}^{(0)} & \longrightarrow & H^{(2g-1)n}(\tilde{V}^{n-1}, ng)_{\text{sgn}_{n-1}}^{(0)} \\ \downarrow \cong & & \downarrow \cong & & \downarrow = \\ H^{(2g-1)n}(\tilde{V}^{n-1}, p_{\tilde{V}^{n-1}}^*\mathcal{H}(ng))_{\text{sgn}_n}^{(0)} & \longrightarrow & H^{(2g-1)n}(\tilde{V}^{n-1}, p_{\tilde{V}^{n-1}}^*\mathcal{L}\text{og}^{(1)}(ng))_{\text{sgn}_n}^{(0)} & \longrightarrow & H^{(2g-1)n}(\tilde{V}^{n-1}, ng)_{\text{sgn}_{n-1}}^{(0)}. \end{array}$$

□

**Corollary 3.6.2.** *The motivic polylogarithm*

$$\text{pol}_{\mathcal{M},\varphi}^n \in H_{\mathcal{M}}^{(2g-1)(n+1)}(V_{\mathcal{A}\backslash\mathcal{A}[N]}^n, (n+1)g)_{\text{sgn}_n}^{(0)}$$

*maps under the regulator maps to the absolute Hodge and to the étale polylogarithm*

$$r_{\mathcal{H}}(\text{pol}_{\mathcal{M},\varphi}^n) = \text{pol}_{\mathcal{H},\varphi}^n \quad r_{\text{et}}(\text{pol}_{\mathcal{M},\varphi}^n) = \text{pol}_{\text{et},\varphi}^n.$$

#### 4. THE ABELIAN POLYLOGARITHM IN DEGREE 0 AND THE HIGHER ANALYTIC TORSION OF THE POINCARÉ BUNDLE

In the first subsection, we recall some concepts and results from Arakelov theory and we state the main result of this section, ie Theorem 4.1.2. In the second subsection, we give a proof of Theorem 4.1.2.

**4.1. Canonical currents on abelian schemes.** We begin with a review of some notations and definitions coming from Arakelov theory.

Let  $(R, \Sigma)$  be an arithmetic ring. By definition, this means that  $R$  is an excellent regular ring, which comes with a finite conjugation-invariant set  $\Sigma$  of embeddings into  $\mathbb{C}$  (see [GS90a, 3.1.2]).

For example  $R$  might be  $\mathbb{Z}$  with its unique embedding into  $\mathbb{C}$ , or  $\mathbb{C}$  with the identity and complex conjugation as embeddings.

An *arithmetic variety* over  $R$  is a regular scheme, which is flat and quasi-projective over  $R$ . This definition is more restrictive than the definition of the same term given in [GS90a].

For any arithmetic variety  $X$  over  $R$ , we write as usual

$$X(\mathbb{C}) := \coprod_{\sigma \in \Sigma} (X \times_{R, \sigma} \mathbb{C})(\mathbb{C}) =: \coprod_{\sigma \in \Sigma} X(\mathbb{C})_{\sigma}.$$

For any  $p \geq 0$ , we let  $D^{p,p}(X_{\mathbb{R}})$  (resp.  $A^{p,p}(X_{\mathbb{R}})$ ) be the  $\mathbb{R}$ -vector space of currents (resp. differential forms)  $\gamma$  on  $X(\mathbb{C})$  such that

- $\gamma$  is a real current (resp. differential form) of type  $(p, p)$ ;
- $F_{\infty}^* \gamma = (-1)^p \gamma$ ,

where  $F_{\infty} : X(\mathbb{C}) \rightarrow X(\mathbb{C})$  is the real analytic involution given by complex conjugation. We then define

$$\tilde{D}^{p,p}(X_{\mathbb{R}}) := D^{p,p}(X_{\mathbb{R}}) / (\text{im } \partial + \text{im } \bar{\partial})$$

(resp.

$$\tilde{A}^{p,p}(X_{\mathbb{R}}) := A^{p,p}(X_{\mathbb{R}}) / (\text{im } \partial + \text{im } \bar{\partial}) \quad ).$$

All these notations are standard in Arakelov geometry. See [Sou92] for a compendium. It is shown in [GS90a, Th. 1.2.2 (ii)] that the natural map  $\tilde{A}^{p,p}(X_{\mathbb{R}}) \rightarrow \tilde{D}^{p,p}(X_{\mathbb{R}})$  is an injection.

If  $Z$  a closed complex submanifold of  $X(\mathbb{C})$ , we shall write more generally  $D_Z^{p,p}(X_{\mathbb{R}})$  for the  $\mathbb{R}$ -vector space of currents  $\gamma$  on  $X(\mathbb{C})$  such that

- $\gamma$  is a real current of type  $(p, p)$ ;
- $F_{\infty}^* \gamma = (-1)^p \gamma$ ;
- the wave-front set of  $\gamma$  is included in  $N_{Z/X(\mathbb{C})} \otimes_{\mathbb{R}} \mathbb{C}$ .

Here  $N$  is the conormal bundle of  $Z$  in  $X(\mathbb{C})$ , where  $Z$  and  $X(\mathbb{C})$  are viewed as real differentiable manifolds.

In the same way as above, we then define the  $\mathbb{R}$ -vector spaces

$$\tilde{D}_Z^{p,p}(X_{\mathbb{R}}) := D_Z^{p,p}(X_{\mathbb{R}})/D_{Z,0}^{p,p}(X_{\mathbb{R}})$$

where  $D_{Z,0}^{p,p}(X_{\mathbb{R}})$  is the set of currents  $\gamma \in D_Z^{p,p}(X_{\mathbb{R}})$  with the following property: there exists a current  $\alpha$  of type  $(p-1, p)$  and a current  $\beta$  of type  $(p, p-1)$  such that  $\gamma := \partial\alpha + \bar{\partial}\beta$  and such that the wave-front sets of  $\alpha$  and  $\beta$  are included in the complexified conormal bundle of  $Z$  in  $X(\mathbb{C})$ .

See [Hör03] for the definition (and theory) of the wave-front set.

It is a consequence of [BGL10, Cor. 4.7] that the natural morphism  $\tilde{D}_Z^{p,p}(X_{\mathbb{R}}) \rightarrow \tilde{D}^{p,p}(X_{\mathbb{R}})$  is an injection.<sup>1</sup>

Furthermore, it is a consequence of [BGL10, Th. 4.3] that for any  $R$ -morphism  $f : Y \rightarrow X$  of arithmetic varieties, there is a natural morphism of  $\mathbb{R}$ -vector spaces

$$f^* : \tilde{D}_Z^{p,p}(X_{\mathbb{R}}) \rightarrow \tilde{D}_{f(\mathbb{C})^*(Z)}^{p,p}(Y_{\mathbb{R}}),$$

provided  $f(\mathbb{C})$  is transverse to  $Z$ . This morphism extends the morphism  $\tilde{A}^{p,p}(X_{\mathbb{R}}) \rightarrow \tilde{A}^{p,p}(Y_{\mathbb{R}})$ , which is obtained by pulling back differential forms.

We shall write  $H_{D,\text{an}}^*(X, \mathbb{R}(\cdot))$  for the analytic real Deligne cohomology of  $X(\mathbb{C})$ . By definition,

$$H_{D,\text{an}}^q(X, \mathbb{R}(p)) := H^q(X(\mathbb{C}), \mathbb{R}(p)_{D,\text{an}})$$

where  $\mathbb{R}(p)_{D,\text{an}}$  is the complex of sheaves of  $\mathbb{R}$ -vector spaces

$$0 \rightarrow \mathbb{R}(p) \rightarrow \Omega_{X(\mathbb{C})}^1 \xrightarrow{d} \Omega_{X(\mathbb{C})}^1 \rightarrow \cdots \rightarrow \Omega_{X(\mathbb{C})}^{p-1} \rightarrow 0$$

on  $X(\mathbb{C})$  (for the ordinary topology). Here  $\mathbb{R}(p) := (2i\pi)^p \mathbb{R} \subseteq \mathbb{C}$  and  $\Omega_{X(\mathbb{C})}^i$  is the sheaf of holomorphic forms of degree  $i$ . Notice that there is morphism of complexes  $\mathbb{R}(p)_{D,\text{an}} \rightarrow \mathbb{R}(p)$ , where  $\mathbb{R}(p)$  is viewed as a complex of sheaves with a single entry in degree 0. This morphism of complexes induces maps

$$\phi_B : H_{D,\text{an}}^*(X, \mathbb{R}(\cdot)) \rightarrow H^*(X(\mathbb{C}), \mathbb{R}(\cdot))$$

from analytic Deligne cohomology into Betti cohomology. We now define

$$H_{D,\text{an}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) := \{\gamma \in H_{D,\text{an}}^{2p-1}(X, \mathbb{R}(p)) \mid F_{\infty}^* \gamma = (-1)^p \gamma\}.$$

It is shown in [BW98, par 6.1] that there is a natural inclusion

$$(4.1.1) \quad H_{D,\text{an}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \hookrightarrow \tilde{A}^{p,p}(X_{\mathbb{R}}).$$

There is a cycle class map  $\text{cyc}_{\text{an}} : H_{\mathcal{M}}^*(X, \bullet) \rightarrow H_{D,\text{an}}^*(X_{\mathbb{R}}, \mathbb{R}(\bullet))$ . The composition  $\phi_{\text{dR}} \circ \text{cyc}_{\text{an}}$  is the usual cycle class map (or "regulator") into Betti cohomology. Finally there is a canonical exact sequence

$$(4.1.2) \quad H_{\mathcal{M}}^{2p-1}(\mathcal{A}, p)_{\mathbb{Q}} \xrightarrow{\text{cyc}_{\text{an}}} \tilde{A}^{p-1,p-1}(\mathcal{A}_{\mathbb{R}}) \rightarrow \widehat{\text{CH}}^p(\mathcal{A})_{\mathbb{Q}} \rightarrow \text{CH}^p(\mathcal{A})_{\mathbb{Q}} \rightarrow 0$$

where (abusing language) the map  $\text{cyc}_{\text{an}}$  is defined via the inclusion 4.1.1. The group  $\text{CH}^p(\mathcal{A})$  is the  $p$ -th Chow group of  $X$  (see the book [Ful98]) and  $\widehat{\text{CH}}^{\bullet}(\cdot)$  is the arithmetic Chow group of

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<sup>1</sup>many thanks to J.-I. Burgos for bringing this to our attention

Gillet-Soulé (see [GS90a]). The group  $\widehat{\text{CH}}^\bullet(\cdot)$  is contravariantly functorial for any  $R$ -morphisms of arithmetic varieties and covariantly functorial for smooth projective morphisms.

Let now as before  $\pi_{\mathcal{A}} = \pi : \mathcal{A} \rightarrow S$  be an abelian scheme over  $S$  of relative dimension  $g = g_{\mathcal{A}}$ . We shall write as usual  $\mathcal{A}^\vee \rightarrow S$  for the dual abelian scheme. We let as before  $\epsilon_{\mathcal{A}} = \epsilon$  be the zero-section of  $\mathcal{A} \rightarrow S$ . We shall denote by  $S_0 = S_{0,\mathcal{A}} = \epsilon_{\mathcal{A}}(S)$  the (reduced) closed subscheme of  $\mathcal{A}$ , which is the image of  $\epsilon_{\mathcal{A}}$ . We write  $\mathcal{P}$  for the Poincaré bundle on  $\mathcal{A} \times_S \mathcal{A}^\vee$ . We endow  $\mathcal{P}(\mathbb{C})$  with the unique metric  $h_{\mathcal{P}}$  such that the canonical rigidification of  $\mathcal{P}$  along the zero-section  $\mathcal{A}^\vee \rightarrow \mathcal{A} \times_S \mathcal{A}^\vee$  is an isometry and such that the curvature form of  $h_{\mathcal{P}}$  is translation invariant along the fibres of the map  $\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} \mathcal{A}^\vee(\mathbb{C}) \rightarrow \mathcal{A}^\vee(\mathbb{C})$ . We write  $\overline{\mathcal{P}} := (\mathcal{P}, h_{\mathcal{P}})$  for the resulting hermitian line bundle. See [MB85, chap I, 4 and chap. I] for more details on all this.

The following is Th. 1.1 in [MR].

**Theorem 4.1.1** (Maillot-R.). *There is a unique class of currents  $\mathfrak{g}_{\mathcal{A}^\vee} \in \widetilde{D}^{g-1, g-1}(\mathcal{A}_{\mathbb{R}})$  with the following three properties:*

- (a) *Any element of  $\mathfrak{g}_{\mathcal{A}^\vee}$  is a Green current for  $S_0(\mathbb{C})$ .*
- (b) *The identity  $(S_0, \mathfrak{g}_{\mathcal{A}^\vee}) = (-1)^g p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}$  holds in  $\widehat{\text{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ .*
- (c) *The identity  $\mathfrak{g}_{\mathcal{A}^\vee} = [n]_* \mathfrak{g}_{\mathcal{A}^\vee}$  holds for all  $n \geq 2$ .*

Here the morphism  $p_1$  is the first projection  $\mathcal{A} \times_S \mathcal{A}^\vee \rightarrow \mathcal{A}$  and  $[n] : \mathcal{A} \rightarrow \mathcal{A}$  is the multiplication-by- $n$  morphism. See [GS90a, 1.2] for the notion of Green current. The symbol  $\widehat{\text{ch}}(\bullet)$  refers to the arithmetic Chern character, which has values in arithmetic Chow groups; see [GS90b] for this. By  $(\cdot)^{(g)}$  we mean the degree  $g$  part of  $(\cdot)$  in the natural grading of the arithmetic Chow group.

In [MR] it is also shown that the restriction  $(-1)^{g+1} \mathfrak{g}_{\mathcal{A}^\vee}|_{\mathcal{A}(\mathbb{C}) \setminus \epsilon(\mathbb{C})}$  is equal to the part of degree  $(g-1, g-1)$  of the Bismut-Koeher higher analytic torsion of  $\overline{\mathcal{P}}$  along the map  $\mathcal{A}(\mathbb{C}) \times \mathcal{A}^\vee(\mathbb{C}) \rightarrow \mathcal{A}(\mathbb{C})$ , for some natural choices of Kähler fibration structures on  $\mathcal{A}(\mathbb{C}) \times \mathcal{A}^\vee(\mathbb{C})$ .

Furthermore (see [MR, Introduction]), the following is known:

- The class of currents  $\mathfrak{g}_{\mathcal{A}^\vee}$  lies in  $\widetilde{D}_{S_0(\mathbb{C})}^{g-1, g-1}(\mathcal{A}_{\mathbb{R}})$ .
- Let  $T$  be an arithmetic variety over  $R$  and  $T \rightarrow S$  be a morphism of schemes over  $R$ . Let  $\mathcal{A}_T$  be the abelian scheme obtained by base-change and let  $\text{BC} : \mathcal{A}_T \rightarrow \mathcal{A}$  be the corresponding morphism. Then  $\text{BC}(\mathbb{C})$  is tranverse to  $S_0(\mathbb{C})$  and  $\text{BC}^* \mathfrak{g}_{\mathcal{A}}^\vee = \mathfrak{g}_{\mathcal{A}_T}^\vee$ .

We now suppose that the field  $k$  is embeddable into  $\mathbb{C}$  and we set  $R = k$ . We choose an arbitrary conjugation-invariant set of embeddings of  $R$  into  $\mathbb{C}$ .

Fix once and for all  $a \in \mathbb{Z}$  such that  $(a, N) = 1$ . Recall that by Corollary 2.2.2, we have an isomorphism

$$(4.1.3) \quad \rho_{\mathcal{A}} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0)^{(0)}$$

and that the element  $\text{pol}_{\mathcal{M}, 1_N^\circ, \mathcal{A}, a}^0$  is the unique element of  $H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)}$  mapping to  $1_N^\circ$  under this isomorphism. Recall that  $1_N^\circ \in H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0)$  is the element given by the formal



sum of all the irreducible components of  $\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S)$ . Recall also that we have

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} := \{h \in H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) \mid \exists l \geq 1 : (\mathrm{tr}_{[a]} - \mathrm{Id})^l(h) = 0\}$$

where  $\mathrm{tr}_{[a]}$  is described in Definition 2.1.1, and

$$H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0)^{(0)} := \{h \in H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0) \mid [a]_*(h) = h\}.$$

**Theorem 4.1.2.** *We have*

$$-2 \cdot \mathrm{cyc}_{\mathrm{an}}(\mathrm{pol}_{\mathcal{M}, 1_N^{\circ}, \mathcal{A}, a}^0) = ([N]^* \mathfrak{g}_{\mathcal{A}^{\vee}} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^{\vee}})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

Furthermore, the map

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \rightarrow H_{D, \mathrm{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

induced by  $\mathrm{cyc}_{\mathrm{an}}$  is injective.

(Theorem 4.1.2 is identical to Theorem 1 in the introduction)

The proof of Theorem 4.1.2 will occupy the next subsection.

The proof goes as follows. We first prove the basic fact that

$$([N]^* \mathfrak{g}_{\mathcal{A}^{\vee}} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^{\vee}})|_{\mathcal{A} \setminus \mathcal{A}[N]} \in \mathrm{cyc}_{\mathrm{an}}(H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)})$$

(see Lemma 4.2.5). This is where Arakelov theory and the geometry of the Poincaré bundle play an important role. Next we show how the elements  $\mathrm{pol}_{\mathcal{M}, 1_N^{\circ}, \mathcal{A}, a}^0$  and  $([N]^* \mathfrak{g}_{\mathcal{A}^{\vee}} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^{\vee}})|_{\mathcal{A} \setminus \mathcal{A}[N]}$  behave under products (see Lemmata 4.2.8 and 4.2.7). To determine the behaviour of  $\mathrm{pol}_{\mathcal{M}, 1_N^{\circ}, \mathcal{A}, a}^0$  under products, we need some elementary invariance results of residue maps in motivic cohomology and to determine the behaviour of  $([N]^* \mathfrak{g}_{\mathcal{A}^{\vee}} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^{\vee}})|_{\mathcal{A} \setminus \mathcal{A}[N]}$  under products, we need parts of the Gillet-Soulé calculus of Green currents. In our third step, we prove that Theorem 4.1.2 holds when  $\mathcal{A}$  is an elliptic curve; this follows from some classical results in the theory of elliptic units (see Lemma 4.2.9). In our fourth and final step, we show that Theorem 4.1.2 holds for products of elliptic curves (see Lemma 4.2.9) and we use a deformation argument together with the existence of moduli spaces for polarised abelian varieties to reduce the general case to the case of products of elliptic curves (see subsubsection 4.2.3).

**Remark 4.1.3.** (important) At first sight, it might seem that a natural way to tackle a statement like Theorem 4.1.2 is to check that the elements  $-2 \cdot \mathrm{cyc}_{\mathrm{an}}(\mathrm{pol}_{\mathcal{M}, 1_N^{\circ}, \mathcal{A}, a}^0)$  and  $([N]^* \mathfrak{g}_{\mathcal{A}^{\vee}} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^{\vee}})|_{\mathcal{A} \setminus \mathcal{A}[N]}$  have the same residue and are both norm invariant (ie  $\mathrm{tr}_{[a]}$ -invariant). One could then argue that a norm invariant element is completely determined by its residue to conclude. This line of proof does not work because there are no localisation sequences in analytic Deligne cohomology (unlike in Deligne-Beilinson cohomology). This is why we have to resort to the more complicated deformation argument outlined above, together with some explicit computations.

**4.2. Proof of Theorem 4.1.2.** We shall suppose without restriction of generality that  $S$  is connected. We also suppose without restriction of generality that  $a \equiv 1 \pmod{N}$ . To see that this last hypothesis does not restrict generality, notice first that  $\text{pol}_{\mathcal{M}, 1_N^\circ, \mathcal{A}, a} = \text{pol}_{\mathcal{M}, 1_N^\circ, \mathcal{A}, a^l}$  for all  $l \geq 1$ . Thus we may replace  $a$  by some  $a^l$  and since  $(\mathbb{Z}/N\mathbb{Z})^*$  is finite there exists  $l \geq 1$  such that  $a^l \equiv 1 \pmod{N}$ .

**4.2.1. Invariance properties of residue maps in motivic cohomology.** In this paragraph, we shall state certain elementary invariance properties of residue maps (see below for the definition of this term) in motivic cohomology. These invariance properties play a key role in the proof of Theorem 4.1.2.

**Proposition 4.2.1.** *Let*

$$\begin{array}{ccc} X & \xhookrightarrow{i} & Y \\ f \uparrow & & \uparrow g \\ X_0 & \xhookrightarrow{l} & Y_0 \end{array}$$

*be a cartesian diagram of smooth schemes over a field. Suppose that the horizontal morphisms are closed immersions. Let  $c \in \mathbb{N}$  and suppose that the codimension of  $X$  in  $Y$  (resp.  $X_0$  in  $Y_0$ ) is  $c$ . Let  $U := Y \setminus X$  and  $U_0 := Y_0 \setminus X_0$ . Then we have a commutative diagram of localisation sequences*

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_{\mathcal{M}}^{\bullet-2c}(X, * - c) & \longrightarrow & H_{\mathcal{M}}^{\bullet}(Y, *) & \longrightarrow & H_{\mathcal{M}}^{\bullet}(U, *) \longrightarrow H_{\mathcal{M}}^{\bullet+1-2c}(X, * - c) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H_{\mathcal{M}}^{\bullet-2c}(X_0, * - c) & \longrightarrow & H_{\mathcal{M}}^{\bullet}(Y_0, *) & \longrightarrow & H_{\mathcal{M}}^{\bullet}(U_0, *) \longrightarrow H_{\mathcal{M}}^{\bullet+1-2c}(X_0, * - c) \longrightarrow \dots \end{array}$$

*where the vertical maps are the pull-back maps.*

*Proof.* See [Dég04]. □

**Proposition 4.2.2.** *Let  $Y$  be a smooth scheme over  $k$  and let  $i : X \hookrightarrow Y$  be a smooth closed subscheme. Let  $X_0$  be a smooth closed subscheme of  $X$ . Let  $c_X, c_Y \in \mathbb{N}$ . Suppose that the codimension of  $X_0$  in  $X$  (resp. in  $Y$ ) is  $c_X$  (resp.  $c_Y$ ). Then the diagram*

$$\begin{array}{ccc} H_{\mathcal{M}}^{\bullet}(X \setminus X_0, *) & \longrightarrow & H_{\mathcal{M}}^{\bullet+1-2c_X}(X_0, * - c_X) \\ \downarrow i_* & & \downarrow = \\ H_{\mathcal{M}}^{\bullet+2(c_Y-c_X)}(Y \setminus X_0, * + c_Y - c_X) & \longrightarrow & H_{\mathcal{M}}^{\bullet+1-2c_X}(X_0, * - c_X) \end{array}$$

*is commutative. Here the horizontal maps are the residue maps.*

*Proof.* We leave this as an exercise for the reader. See [Jan90, par. 6]. □

Here is how Proposition 4.2.1 applies in our setting.

Let  $T$  be an arithmetic variety over  $R = k$  and  $T \rightarrow S$  be a morphism of schemes over  $R$ . Let  $\mathcal{A}_T$  be the abelian scheme obtained by base-change and let  $\text{BC} : \mathcal{A}_T \rightarrow \mathcal{A}$  be the corresponding morphism.

Let

$$\text{BC}_l : \mathcal{A}_T[N] \setminus \epsilon_{\mathcal{A}_T}(S) \rightarrow \mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S)$$

be the morphism obtained by restricting  $\text{BC}$ . Similarly, let

$$\text{BC}_h : \mathcal{A}_T \setminus \mathcal{A}_T[N] \rightarrow \mathcal{A} \setminus \mathcal{A}[N]$$

be the morphism obtained by restricting  $\text{BC}$ .

**Lemma 4.2.3.** *We have  $\text{BC}_l^* \circ \rho_{\mathcal{A}} = \rho_{\mathcal{A}_T} \circ \text{BC}_h^*$ .*

*Proof.* Follows from 4.2.1. □

4.2.2. *An intermediate result.*

**Proposition 4.2.4.** (a) *The map*

$$H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \rightarrow H_{D, \text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

*induced by  $\text{cyc}_{\text{an}}$  is injective.*

(b) *Let  $\sigma \in \mathcal{A}[N](S) \setminus \epsilon_{\mathcal{A}}(S)$ . Let  $\text{pol}_{-\sigma} \in H_{\mathcal{M}}^{2g_{\mathcal{A}}-1}(\mathcal{A} \setminus \mathcal{A}[N], g_{\mathcal{A}})^{(0)}$  be the element corresponding to the class of  $-\sigma$  in  $H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0)$ . Then*

$$-2 \cdot \text{cyc}_{\text{an}}(\text{pol}_{-\sigma}) = (\sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}.$$

The proof of Proposition 4.2.4 will rely on the following lemmata.

**Lemma 4.2.5.** *Let  $\sigma \in \mathcal{A}[N](S) \setminus \epsilon_{\mathcal{A}}(S)$ . Then*

$$(\sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \{\epsilon_{\mathcal{A}}(S), -\sigma(S)\}} \subseteq \text{cyc}_{\text{an}}(H_{\mathcal{M}}^{2g_{\mathcal{A}}-1}(\mathcal{A} \setminus \{\epsilon_{\mathcal{A}}(S), -\sigma(S)\}, g_{\mathcal{A}}))^{(0)}$$

*Proof.* It is sufficient to show that

$$(4.2.1) \quad (\sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \{\epsilon_{\mathcal{A}}(S), -\sigma(S)\}} \subseteq \text{cyc}_{\text{an}}(H_{\mathcal{M}}^{2g_{\mathcal{A}}-1}(\mathcal{A} \setminus \mathcal{A}[N], g_{\mathcal{A}})).$$

Indeed, a natural analog of the operator  $\text{tr}_{[a]}$  operates on analytic Deligne cohomology and the map  $\text{cyc}_{\text{an}}$  intertwines this operator with  $\text{tr}_{[a]}$ . So if 4.2.1 holds, we may deduce the conclusion of the lemma from the existence of the Jordan decomposition.

Let  $\Sigma$  be the rigidified line bundle on  $\mathcal{A}^\vee$  corresponding to  $\sigma$  via the  $S$ -isomorphism  $\mathcal{A} \simeq (\mathcal{A}^\vee)^\vee$ . Let  $\bar{\Sigma}$  be the unique hermitian line bundle on  $\mathcal{A}^\vee$ , such that

- its underlying line bundle is  $\Sigma$ ;
- the rigidification is an isometry;

- its curvature form is translation invariant on the fibres of the map  $\mathcal{A}^\vee(\mathbb{C}) \rightarrow S(\mathbb{C})$ .

To prove relation 4.2.5, we compute in  $\widehat{\text{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ :

$$\begin{aligned} \sigma^*(S_0, \mathfrak{g}_{\mathcal{A}^\vee}) - (S_0, \mathfrak{g}_{\mathcal{A}^\vee}) &= (-1)^g \left( \sigma^*(p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)}) - p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)} \right) \\ &= (-1)^g \left( p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}} \otimes \overline{\Sigma}))^{(g)} - p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)} \right) \\ &= (-1)^g \left( p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)} - p_{1,*}(\widehat{\text{ch}}(\overline{\mathcal{P}}))^{(g)} \right) = 0 \end{aligned}$$

Here we used in the second equality the fact that the direct image in arithmetic Chow theory is naturally compatible with smooth base-change. We also used the fact that  $\widehat{\text{ch}}(\overline{\Sigma}) = 1$  and the multiplicativity of the arithmetic Chern character. Now, we have

$$\sigma^*(S_0, \mathfrak{g}_{\mathcal{A}^\vee}) - (S_0, \mathfrak{g}_{\mathcal{A}^\vee}) = (-\sigma(S), \sigma^* \mathfrak{g}_{\mathcal{A}^\vee}) - (S_0, \mathfrak{g}_{\mathcal{A}^\vee})$$

and thus the image of

$$(\sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \{\epsilon_{\mathcal{A}}(S), -\sigma(S)\}}$$

in  $\widehat{\text{CH}}^g(\mathcal{A} \setminus \{\epsilon_{\mathcal{A}}(S), -\sigma(S)\})_{\mathbb{Q}}$  vanishes. Using 4.1.2, we may conclude.  $\square$

The next lemma proves assertion (a) in Proposition 4.2.4.

**Lemma 4.2.6.** *The map  $H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} \rightarrow H_{D, \text{an}}^{2g-1}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$  induced by  $\text{cyc}_{\text{an}}$  is injective.*

*Proof.* Notice that we have commutative diagram

$$\begin{array}{ccc} H_{\mathcal{M}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)} & \xrightarrow{\rho_{\mathcal{A}}, \cong} & H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0) \\ \downarrow \text{cyc}_{\text{an}} & & \downarrow \text{cyc}_{\text{an}} \\ H_{D, \text{an}}^{2g-1}(\mathcal{A} \setminus \mathcal{A}[N], g) & & H_{D, \text{an}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0) \\ \downarrow \phi_B & & \downarrow \phi_B, \cong \\ H^{2g-1}(\mathcal{A}(\mathbb{C}) \setminus \mathcal{A}[N](\mathbb{C}), g) & \xrightarrow{\cong} & H^0(\mathcal{A}[N](\mathbb{C}) \setminus \epsilon_{\mathcal{A}}(S)(\mathbb{C}), 0) \end{array}$$

where the isomorphism on the bottom line is the residue map in Betti cohomology. The fact that the residue map in Betti cohomology is an isomorphism follows from the fact that Betti cohomology is a twisted Poincaré duality theory in the sense of Bloch-Ogus and Corollary 2.2.2. Furthermore, the upper right vertical map can be seen to be injective. This implies the assertion of the Lemma.  $\square$

**Lemma 4.2.7.** *Let  $\mathcal{B} \rightarrow S$  be an abelian scheme.*

*Let  $\sigma \in \mathcal{A}[N](S)$  and let  $\tau \in \mathcal{B}[N](S)$ .*

*Let  $x := (\sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \setminus \mathcal{A}[N]}$  and  $y := (\tau^* \mathfrak{g}_{\mathcal{B}^\vee} - \mathfrak{g}_{\mathcal{B}^\vee})|_{\mathcal{B} \setminus \mathcal{B}[N]}$ .*

*Furthermore, let*

$$z := ((\sigma \times \tau)^* \mathfrak{g}_{\mathcal{A}^\vee \times \mathcal{B}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee \times \mathcal{B}^\vee})|_{\mathcal{A} \times \mathcal{B} \setminus (\mathcal{A} \times \mathcal{B})[N]}.$$

Then we have

$$z = (\text{Id}_{\mathcal{A} \setminus \mathcal{A}[N]} \times \tau)_* x + (0 \times \text{Id}_{\mathcal{B} \setminus \mathcal{B}[N]})_* y.$$

*Proof.* Let  $\nu_{\mathcal{A}^\vee} := (-1)^g p_{1,*}(\text{ch}(\overline{\mathcal{P}}))^{(g)}$ . We shall write  $\delta_{\mathcal{A} \times 0}$  for the Dirac current in  $\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} \mathcal{B}(\mathbb{C})$  of the closed complex manifold associated with the image of  $\mathcal{A}$  by the morphism  $(\text{Id}_{\mathcal{A}} \times_S \epsilon_{\mathcal{B}} \circ \pi_{\mathcal{A}})$ . We shall also use the notation  $\delta_{\mathcal{A} \times \tau}$ , which is defined similarly. Let  $q_{\mathcal{A}} : \mathcal{A} \times_S \mathcal{B} \rightarrow \mathcal{A}$  and  $q_{\mathcal{B}} : \mathcal{A} \times_S \mathcal{B} \rightarrow \mathcal{B}$  be the obvious projections. First we make the computation

$$\begin{aligned} & q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} \wedge (\delta_{\mathcal{A} \times 0} - \delta_{\mathcal{A} \times \tau}) + (q_{\mathcal{B}}^* \tau^* \mathfrak{g}_{\mathcal{B}^\vee} - q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee}) \wedge (\delta_{0 \times \mathcal{B}} - q_{\mathcal{A}}^* \nu_{\mathcal{A}^\vee}) \\ &= q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} \wedge (\delta_{\mathcal{A} \times 0} - \delta_{\mathcal{A} \times \tau}) - (q_{\mathcal{B}}^* \tau^* \mathfrak{g}_{\mathcal{B}^\vee} - q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee}) \wedge (\text{dd}^c q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee}) \\ &= (\delta_{\mathcal{A} \times 0} - \delta_{\mathcal{A} \times \tau} + \delta_{\mathcal{A} \times \tau} - q_{\mathcal{B}}^* \tau^* \nu_{\mathcal{B}^\vee} - \delta_{\mathcal{A} \times 0} + q_{\mathcal{B}}^* \nu_{\mathcal{B}^\vee}) \wedge q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} = 0 \end{aligned}$$

In this computation, we used the following elementary fact (see [Roe99, before par. 6.2.1]): if  $\eta, \omega$  are two currents of type  $(p, p)$  on a complex manifold, such that  $\eta$  and  $\omega$  have disjoint wave front sets, then

$$\eta \wedge \text{dd}^c \omega - \text{dd}^c \eta \wedge \omega \in \text{im} \partial + \text{im} \bar{\partial}.$$

Now using the formula [MR, Th. 1.2, 5.] for the canonical current of fibre products of abelian schemes and the previous computation, we get that

$$\begin{aligned} & (\sigma \times \tau)^* \mathfrak{g}_{\mathcal{A}^\vee \times \mathcal{B}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee \times \mathcal{B}^\vee} \\ &= (\sigma \times \tau)^* (q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} * q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee}) - \mathfrak{g}_{\mathcal{A}^\vee \times \mathcal{B}^\vee} = q_{\mathcal{A}}^* \sigma^* \mathfrak{g}_{\mathcal{A}^\vee} * q_{\mathcal{B}}^* \tau^* \mathfrak{g}_{\mathcal{B}^\vee} - q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} * q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee} \\ &= q_{\mathcal{A}}^* \sigma^* \mathfrak{g}_{\mathcal{A}^\vee} \wedge \delta_{\mathcal{A} \times \tau} + q_{\mathcal{A}}^* \sigma^* \nu_{\mathcal{A}^\vee} \wedge q_{\mathcal{B}}^* \tau^* \mathfrak{g}_{\mathcal{B}^\vee} - q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} \wedge \delta_{\mathcal{A} \times 0} - q_{\mathcal{A}}^* \nu_{\mathcal{A}^\vee} \wedge q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee} \\ &= q_{\mathcal{A}}^* \sigma^* \mathfrak{g}_{\mathcal{A}^\vee} \wedge \delta_{\mathcal{A} \times \tau} + q_{\mathcal{A}}^* \sigma^* \nu_{\mathcal{A}^\vee} \wedge q_{\mathcal{B}}^* \tau^* \mathfrak{g}_{\mathcal{B}^\vee} - q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} \wedge \delta_{\mathcal{A} \times 0} - q_{\mathcal{A}}^* \nu_{\mathcal{A}^\vee} \wedge q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee} \\ &+ q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee} \wedge (\delta_{\mathcal{A} \times 0} - \delta_{\mathcal{A} \times \tau}) + (q_{\mathcal{B}}^* \tau^* \mathfrak{g}_{\mathcal{B}^\vee} - q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee}) \wedge (\delta_{0 \times \mathcal{B}} - q_{\mathcal{A}}^* \nu_{\mathcal{A}^\vee}) \\ &= (q_{\mathcal{A}}^* \sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - q_{\mathcal{A}}^* \mathfrak{g}_{\mathcal{A}^\vee}) \wedge \delta_{\mathcal{A} \times \tau} + (q_{\mathcal{B}}^* \tau^* \mathfrak{g}_{\mathcal{B}^\vee} - q_{\mathcal{B}}^* \mathfrak{g}_{\mathcal{B}^\vee}) \wedge \delta_{0 \times \mathcal{B}} \end{aligned}$$

which implies the assertion of the lemma.  $\square$

**Lemma 4.2.8.** *Let  $\mathcal{B} \rightarrow S$  be an abelian scheme of relative dimension  $g_{\mathcal{B}}$ . Let  $\sigma \in \mathcal{A}[N](S) \setminus \epsilon_{\mathcal{A}}(S)$  and let  $\tau \in \mathcal{B}[N](S) \setminus \epsilon_{\mathcal{B}}(S)$ .*

*Let  $x \in H_{\mathcal{M}}^{2g_{\mathcal{A}}-1}(\mathcal{A} \setminus \mathcal{A}[N], g_{\mathcal{A}})^{(0)}$  be the element corresponding to the class of  $\sigma$  in  $H_{\mathcal{M}}^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0)$ .*

*Let  $y \in H_{\mathcal{M}}^{2g_{\mathcal{B}}-1}(\mathcal{B} \setminus \mathcal{B}[N], g_{\mathcal{B}})^{(0)}$  be the element corresponding to the class of  $\tau$  in  $H_{\mathcal{M}}^0(\mathcal{B}[N] \setminus \epsilon_{\mathcal{B}}(S), 0)$ .*

*Then  $z := (\text{Id}_{\mathcal{A} \setminus \mathcal{A}[N]} \times \tau)_* x + (0 \times \text{Id}_{\mathcal{B} \setminus \mathcal{B}[N]})_* y$  lies in  $H_{\mathcal{M}}^{2(g_{\mathcal{A}}+g_{\mathcal{B}})-1}(\mathcal{A} \times_S \mathcal{B} \setminus (\mathcal{A} \times_S \mathcal{B})[N], g_{\mathcal{A}} + g_{\mathcal{B}})^{(0)}$  and corresponds to the class of  $\sigma \times \tau$  in  $H_{\mathcal{M}}^0((\mathcal{A} \times_S \mathcal{B})[N] \setminus \epsilon_{\mathcal{A} \times_S \mathcal{B}}(S), 0)$ .*

*Proof.* For the first assertion, it is sufficient to show that

$$z \in H_{\mathcal{M}}^{2(g_{\mathcal{A}}+g_{\mathcal{B}})-1}(\mathcal{A} \times_S \mathcal{B} \setminus (\mathcal{A} \times_S \mathcal{B})[N], g_{\mathcal{A}} + g_{\mathcal{B}})^{(0)}$$

and that  $z$  has residue  $\sigma \times \tau$  in  $H_{\mathcal{M}}^0((\mathcal{A} \times \mathcal{B})[N] \setminus \epsilon_{\mathcal{A} \times \mathcal{B}}(S), 0)$ .

The fact that

$$z \in H_{\mathcal{M}}^{2(g_{\mathcal{A}}+g_{\mathcal{B}})-1}(\mathcal{A} \times_S \mathcal{B} \setminus (\mathcal{A} \times_S \mathcal{B})[N], g_{\mathcal{A}} + g_{\mathcal{B}}))^{(0)}$$

follows from the fact that the morphisms  $\text{Id}_{\mathcal{A} \setminus \mathcal{A}[N]} \times \tau$  and  $0 \times \text{Id}_{\mathcal{B} \setminus \mathcal{B}[N]}$  commute with the multiplication-by- $a$  morphism (since  $a \equiv 1 \pmod{N}$ ).

To compute the residue of  $z$ , notice first that the value of the residue of  $x$  (resp.  $y$ ) at  $\epsilon_{\mathcal{A}}(S)$  (resp.  $\epsilon_{\mathcal{B}}(S)$ ) is  $-1$ . To see this, notice that the push-forward into  $H_{\mathcal{M}}^{2g_{\mathcal{A}}}(\mathcal{A}, g_{\mathcal{A}})$  (resp. into  $H_{\mathcal{M}}^{2g_{\mathcal{B}}}(\mathcal{B}, g_{\mathcal{B}})$ ) of the residue of  $x$  (resp.  $y$ ) in  $H^0(\mathcal{A}[N], 0)$  (resp.  $H^0(\mathcal{B}[N], 0)$ ) vanishes. This vanishing implies that the push-forward to  $H^0(S, 0) \simeq \mathbb{Q}$  of the residue of  $x$  (resp.  $y$ ) in  $H^0(\mathcal{A}[N], 0)$  (resp.  $H^0(\mathcal{B}[N], 0)$ ) vanishes too. If one combine this fact with the fact that the residue of  $x$  (resp.  $y$ ) in  $H^0(\mathcal{A}[N] \setminus \epsilon_{\mathcal{A}}(S), 0)$  (resp.  $H^0(\mathcal{B}[N] \setminus \epsilon_{\mathcal{B}}(S), 0)$ ) is  $\sigma(S)$  (resp.  $\tau(S)$ ) (this holds by hypothesis), one obtains that the residue of  $x$  (resp.  $y$ ) at  $\epsilon_{\mathcal{A}}(S)$  (resp.  $\epsilon_{\mathcal{B}}(S)$ ) is  $-1$ .

Now using Proposition 4.2.2, we may compute that the residue of  $z$  in  $H^0((\mathcal{A} \times_S \mathcal{B})[N], 0)$  is

$$(\sigma \times \tau - 0 \times \tau) + (0 \times \tau - 0 \times 0)$$

and thus the residue of  $z$  in  $H_{\mathcal{M}}^0((\mathcal{A} \times \mathcal{B})[N] \setminus \epsilon_{\mathcal{A} \times \mathcal{B}}(S), 0)$  is  $\sigma \times \tau$ .  $\square$

**Lemma 4.2.9.** *Proposition 4.2.4 (b) holds if the morphism  $S \rightarrow \text{Spec } k$  is the identity and  $\mathcal{A} \simeq \prod_{i=1, S}^g E_i$ , where  $E_i$  is an elliptic curve over  $S = \text{Spec } k$ .*

*Proof.* (of Lemma 4.2.9) We first prove the statement when  $g = 1$ . Let  $E := E_1 = E_g$ . Notice that the map

$$\begin{aligned} \text{cyc}_{\text{an}} &: H_{\mathcal{M}}^1(E \setminus E[N], 1) = \mathcal{O}_{E \setminus E[N]}^*(E \setminus E[N]) \otimes \mathbb{Q} \\ &\rightarrow H_{D, \text{an}}^1(E \setminus E[N], 1) = \{f \in C^\infty((E \setminus E[N])(\mathbb{C}), \mathbb{R}) \mid \text{dd}^c f = 0\} \end{aligned}$$

can be explicitly described by the formula  $u \otimes r \mapsto r \log |u(\mathbb{C})|$ .

Now let  $\alpha$  be one of the given embeddings of  $R$  into  $\mathbb{C}$  (those that are part of the datum of an arithmetic ring). Let  $\mathbb{C}/[1, \tau_{E, \alpha}] \simeq E(\mathbb{C})_\alpha$  be a presentation of  $E(\mathbb{C})_\alpha$  as a quotient of  $\mathbb{C}$  by a lattice generated by 1 and a complex number  $\tau_{E, \alpha}$  with strictly positive imaginary part. Call  $\lambda : \mathbb{C} \rightarrow E(\mathbb{C})_\alpha$  the corresponding quotient map. Then by [MR, par. 7], we have

$$\mathfrak{g}_{E^\vee}(\lambda(z)) = -2 \log |e^{-z \cdot \eta(z)/2} \text{sigma}(z) \Delta(\tau_{E, \alpha})^{\frac{1}{12}}|$$

for all  $z \notin [1, \tau_{E, \alpha}]$ . Here  $\Delta(\bullet)$  is the discriminant modular form,  $\text{sigma}(z)$  is the Weierstrass sigma-function associated with the lattice  $[1, \tau_{E, \alpha}]$  and  $\eta$  is the quasi-period map associated with the lattice  $[1, \tau_{E, \alpha}]$ , extended  $\mathbb{R}$ -linearly to all of  $\mathbb{C}$  (see [Sil94, I, Prop. 5.2] for the latter). Let

$z_0 \in \mathbb{C}$  such that  $\lambda(z_0) = -\sigma_\alpha$ . We compute

$$\begin{aligned}
& \left| \frac{e^{-(z-z_0) \cdot \eta(z-z_0)/2} \text{sigma}(z-z_0)}{e^{-z \cdot \eta(z)/2} \text{sigma}(z)} \right| \\
&= \left| \frac{e^{-\frac{1}{2}(z \cdot \eta(z) - z \cdot \eta(z_0) - z_0 \cdot \eta(z) + z_0 \cdot \eta(z_0))} \text{sigma}(z-z_0)}{e^{-z \cdot \eta(z)/2} \text{sigma}(z)} \right| \\
&= \left| e^{\frac{1}{2}(z \cdot \eta(z_0) + z_0 \cdot \eta(z) - z_0 \cdot \eta(z_0))} \frac{\text{sigma}(z-z_0)}{\text{sigma}(z)} \right| \\
&= \left| e^{z \cdot \eta(z_0) - \frac{1}{2} z_0 \cdot \eta(z_0)} \frac{\text{sigma}(z-z_0)}{\text{sigma}(z)} \right|.
\end{aligned}$$

Here we used the Legendre relation for the quasi-period map on the last line (see [Sil94, I, Prop. 5.2, (d)]). Let

$$\phi(z) := e^{z \cdot \eta(z_0) - \frac{1}{2} z_0 \cdot \eta(z_0)} \frac{\text{sigma}(z-z_0)}{\text{sigma}(z)}.$$

Now recall that the periodicity relation for the sigma function says that

$$\text{sigma}(z + \omega) = \psi(\omega) e^{\eta(\omega)(z + \omega/2)} \text{sigma}(z)$$

for all  $\omega \in [1, \tau_{E,\alpha}]$  (see [Sil94, I, Prop. 5.4 (c)]). Here  $\psi(\bullet)$  is a function with values in the set  $\{-1, 1\}$ . This implies that

$$\frac{\text{sigma}(z + \omega - z_0)}{\text{sigma}(z + \omega)} = \frac{\text{sigma}(z - z_0)}{\text{sigma}(z)} e^{-\eta(\omega)z_0}$$

and thus that

$$\phi(z + \omega) / \phi(z) = e^{\omega \cdot \eta(z_0) - \eta(\omega)z_0} := \alpha(\omega, z_0).$$

The Legendre relation again implies that  $\alpha(\bullet, z_0)$  defines a homomorphism of abelian groups  $[1, \tau_{E,\alpha}] \rightarrow \mathbb{C}^*$  and that its image is a torsion group of order dividing  $N$ . We conclude that  $\phi(z)^N$  is a  $[1, \tau_{E,\alpha}]$ -periodic function. Furthermore,  $\phi(z)$  has a zero of order 1 at  $z_0$  and a pole of order 1 at 0. We see that after passage to the quotient, the function  $\phi(z)$  defines an element  $\phi_0 \in \mathcal{O}^*(E(\mathbb{C})_\alpha \backslash E(\mathbb{C})_\alpha[N]) \otimes \mathbb{Q}$ , whose divisor is  $z_0 \ominus 0$ . Furthermore, the distribution relations of A. Robert (see [KL81, par. 4, Th. 4.1]) show that  $\text{tr}_{[a]}(\phi_0) = \phi_0$  (we slightly abuse notation here).

Now let  $\tilde{\phi}_0 \in H^1_{\mathcal{M}}(E \backslash E[N], 1)^{(0)}$  be an element such that

$$\text{cyc}_{\text{an}}(\tilde{\phi}_0)|_{E(\mathbb{C})_\alpha} = \log |\tilde{\phi}_0| = ([-z_0]^* \mathfrak{g}_{E^\vee} - \mathfrak{g}_{E^\vee})|_{E(\mathbb{C})_\alpha \backslash E(\mathbb{C})_\alpha[N]} = -2 \log |\phi_0| = \log |\phi_0^{-2}|$$

This exists by Lemma 4.2.5. Notice that both  $\tilde{\phi}_0$  and  $\phi_0$  are invariant under  $\text{tr}_{[a]}$  and thus  $\frac{\tilde{\phi}_0(\mathbb{C})}{\phi_0^{-2}}$  is a constant  $l \in \mathbb{C}^* \otimes \mathbb{Q}$ , such that  $\text{tr}_{[a]}(l) = l^{a^2} = l$  and thus  $l = 1$ . Thus we may compute that  $\text{div}(\tilde{\phi}_0) = 2 \odot 0 \ominus 2 \odot z_0$ . We conclude that  $\tilde{\phi}_0 = -2 \cdot \text{pol}_{z_0}$  and this concludes the proof of the lemma when  $g = 1$ . To prove the Lemma 4.2.9 in general, just combine the fact that Lemma 4.2.9 holds for  $g = 1$  with Lemmata 4.2.7 and 4.2.8.

□

Let  $T$  be an arithmetic variety over  $R = k$  and  $T \rightarrow S$  be a morphism of schemes over  $R$ . Let  $\mathcal{A}_T$  be the abelian scheme obtained by base-change and let  $\mathrm{BC}, \mathrm{BC}_h$  and  $\mathrm{BC}_l$  be as at the end of paragraph 4.2.1. Let

$$\mathrm{BC}_l^* : H_{\mathcal{M}}^0(\mathcal{A}[N] \backslash \epsilon_{\mathcal{A}}(S), 0) \rightarrow H_{\mathcal{M}}^0(\mathcal{A}_T[N] \backslash \epsilon_{\mathcal{A}_T}(S), 0)$$

and

$$\mathrm{BC}_h^* : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \backslash \mathcal{A}[N], g)^{(0)} \rightarrow H_{\mathcal{M}}^{2g-1}(\mathcal{A}_T \backslash \mathcal{A}_T[N], g)^{(0)}$$

be the natural pull-back maps.

**Lemma 4.2.10.** *If  $T$  is connected and  $\mathcal{A}[N] \simeq (\mathbb{Z}/N\mathbb{Z})_S^{2g}$  then the maps  $\mathrm{BC}_l^*$  and  $\mathrm{BC}_h^*$  are isomorphisms.*

*Proof.* The map  $\mathrm{BC}_l^*$  is an isomorphism by construction and the fact that the map  $\mathrm{BC}_h^*$  is an isomorphism follows from the fact that  $\mathrm{BC}_l^*$  is an isomorphism, from Lemma 4.2.3 and from the isomorphisms

$$\rho_{\mathcal{A}} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \backslash \mathcal{A}[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}[N] \backslash \epsilon_{\mathcal{A}}(S), 0).$$

and

$$\rho_{\mathcal{A}_T} : H_{\mathcal{M}}^{2g-1}(\mathcal{A}_T \backslash \mathcal{A}_T[N], g)^{(0)} \cong H_{\mathcal{M}}^0(\mathcal{A}_T[N] \backslash \epsilon_{\mathcal{A}_T}(S), 0).$$

□

**Lemma 4.2.11.** *Let  $\sigma \in \mathcal{A}[N](S) \backslash \epsilon_{\mathcal{A}}(S)$ . We have*

$$\mathrm{BC}_h^* \mathrm{pol}_{\sigma} = \mathrm{pol}_{\sigma_T}$$

and

$$\mathrm{BC}_h^*(\sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \backslash \mathcal{A}[N]} = (\sigma_T^* \mathfrak{g}_{\mathcal{A}_T^\vee} - \mathfrak{g}_{\mathcal{A}_T^\vee})|_{\mathcal{A}_T \backslash \mathcal{A}_T[N]}.$$

*Proof.* The second equality is a consequence of the remarks after Theorem 4.1.1. The first equality is a consequence of the fact that  $\mathrm{BC}_h^* \circ \mathrm{tr}_{a, \mathcal{A}} = \mathrm{tr}_{a, \mathcal{A}_T} \circ \mathrm{BC}_h^*$  and of Lemma 4.2.3. □

**Lemma 4.2.12.** *Suppose that  $\mathcal{A}[N] \simeq (\mathbb{Z}/N\mathbb{Z})_S^{2g}$ . Suppose also that for some point  $s \in S(k)$ , the abelian scheme  $\mathcal{A}_s$  is a product of elliptic curves over  $k$ . Then Proposition 4.2.4 holds for  $\mathcal{A} \rightarrow S$ .*

*Proof.* Let  $T := \mathrm{Spec} k$  and let  $T \rightarrow S$  be the closed immersion given by  $s$ . By Lemmata 4.2.6 and 4.2.10, the map

$$\mathrm{cyc}_{\mathrm{an}} \circ \mathrm{BC}_h^* = \mathrm{BC}_h^* \circ \mathrm{cyc}_{\mathrm{an}} : H_{\mathcal{M}}^{2g-1}(\mathcal{A} \backslash \mathcal{A}[N], g)^{(0)} \rightarrow H_{D, \mathrm{an}}^{2g-1}((\mathcal{A}_s \backslash \mathcal{A}_s[N])_{\mathbb{R}}, \mathbb{R}(g))$$

is injective. Now let  $\phi \in H_{\mathcal{M}}^{2g-1}(\mathcal{A} \backslash \mathcal{A}[N], g)^{(0)}$  be an element such that

$$-\mathrm{cyc}_{\mathrm{an}}(\phi) = (\sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A} \backslash \mathcal{A}[N]}.$$

The element  $\phi$  exists by Lemma 4.2.5. Then by Lemma 4.2.11 and Lemma 4.2.9, we have

$$-\mathrm{BC}_h^*(\mathrm{cyc}_{\mathrm{an}}(\phi - \mathrm{pol}_{-\sigma})) = 0$$

whence  $\phi = \mathrm{pol}_{-\sigma}$ . □



**Lemma 4.2.13.** *Let  $M \in \mathbb{N}^*$ . To prove Proposition 4.2.4, it is sufficient to prove it under the supplementary assumption that  $\mathcal{A}[M] \simeq (\mathbb{Z}/M\mathbb{Z})_S^{2g}$ .*

*Proof.* We may choose  $T \rightarrow S$ , such that  $T \rightarrow S$  is proper and generically finite and such that  $\mathcal{A}_T[M] \simeq (\mathbb{Z}/M\mathbb{Z})_T^{2g}$ . In view of this as well as Lemma 4.2.11 and Lemma 4.2.5, we see that it is sufficient to show that the map  $\text{BC}_h^*$  is injective when  $T \rightarrow S$  is proper and generically finite. This is a consequence of the fact that  $\text{BC}_h$  is then also proper and generically finite, of the projection formula, and of the fact that  $\text{BC}_{h,*}(1) = \deg(\text{BC}_h)$ .  $\square$

*Proof of Proposition 4.2.4.*

Let  $M \in \mathbb{N}^*$ . Suppose that  $N|M$  and that  $M > 4$ . By Lemma 4.2.13, we may suppose without restriction of generality that  $\mathcal{A}[M] \simeq (\mathbb{Z}/M\mathbb{Z})_S^{2g}$ . Now by Lemma 4.2.11, to prove Proposition 4.2.4, it is sufficient to show that there exists

- $T$ , an arithmetic variety over  $R$ ;
- $\mathcal{B} \rightarrow T$ , an abelian scheme of relative dimension  $g$  such that  $\mathcal{B}[M] \simeq (\mathbb{Z}/M\mathbb{Z})_T^{2g}$  and such that Proposition 4.2.4 holds for  $\mathcal{B}$ ;
- an  $R$ -morphism  $S \rightarrow T$ , such that  $\mathcal{A} \simeq \mathcal{B} \times_T S$ .

Before trying to determine a suitable  $\mathcal{B}$ , equip  $\mathcal{A} \rightarrow S$  with a polarisation and let  $d$  be the degree of the latter. As a candidate for  $\mathcal{B}$ , we consider the universal abelian scheme over the (fine) moduli space  $\text{A}_{g,d,M}/k$  of abelian varieties of dimension  $g$ , equipped with an  $M$ -level structure and a polarisation of degree  $d$  (see for instance [MB85] for more details). Proposition 4.2.4 holds for  $\mathcal{B}$  by Lemma 4.2.12 and there exists a  $k$ -morphism  $S \rightarrow T$ , such  $\mathcal{A} \simeq \mathcal{B} \times_T S$ , because  $\text{A}_{g,d,M}/k$  is a moduli space.

4.2.3. *Proof of Theorem 4.1.2.*

**Lemma 4.2.14.** *Let  $M \in \mathbb{N}^*$ . To prove Theorem 4.1.2, it is sufficient to prove it under the supplementary assumption that  $\mathcal{A}[M] \simeq (\mathbb{Z}/M\mathbb{Z})_S^{2g}$ .*

*Proof.* The proof is similar to the proof of Lemma 4.2.13 and will be omitted.  $\square$

**Lemma 4.2.15.** *Suppose that  $\mathcal{A}[N] \simeq (\mathbb{Z}/N\mathbb{Z})_S^{2g}$ . For any  $\sigma \in \mathcal{A}[N](S)$ , let*

$$x_\sigma := \sigma^* \mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}_{\mathcal{A}^\vee}.$$

*Then we have*

$$\sum_{\sigma} x_\sigma = [N]^* \mathfrak{g}_{\mathcal{A}^\vee} - N^{2g} \cdot \mathfrak{g}_{\mathcal{A}^\vee}.$$

*Proof.* If  $\eta$  is a differential form on  $\mathcal{A}(\mathbb{C})$ , we have from the definitions that

$$[N]^*[N]_*\eta = \sum_{\sigma} \sigma_*\eta$$

Dualising this statement for currents, we obtain that

$$[N]^*[N]_*\mathfrak{g}_{\mathcal{A}^\vee} = \sum_{\sigma} \sigma^* \mathfrak{g}_{\mathcal{A}^\vee}.$$

Since  $[N]_*\mathfrak{g}_{\mathcal{A}^\vee} = \mathfrak{g}_{\mathcal{A}^\vee}$  by 4.1.1 (c), we obtain that

$$[N]^*\mathfrak{g}_{\mathcal{A}^\vee} = \sum_{\sigma} \sigma^* \mathfrak{g}_{\mathcal{A}^\vee}$$

from which the Lemma follows.  $\square$

The proof of Theorem 4.1.2 now follows from Lemma 4.2.14, Proposition 4.2.4 and Lemma 4.2.15.

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